

SCATTERING OF ELASTIC WAVES BY COMPLEX DEFECTS;
MULTIPLE SCATTERING FORMALISM

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ABSTRACT

Scattering of elastic waves by complex defects is investigated. In previous work the Distorted Wave Born Approximation was developed, in which a general shaped defect is represented as a sphere S and a remainder volume R , the latter being treated in first order perturbation theory. The DWBA was used to study scattering by a sequence of defects representing increasingly larger deviations from spherical; comparisons with experiment and other theories were made. This approach breaks down when scattering by R is of similar magnitude as by S . Such is the case for complex defects such as two spheres, or a sphere and a crack emanating from it. To treat such situations, a multiple scattering formalism was developed.

INTRODUCTION

Development of approximate solutions to the scattering of elastic waves by defects has proved to be most useful for progress in nondestructive evaluation of materials. Applications of such approximate methods include comparisons with experiment, identification of features sensitive to scatterer characteristics of special interest, incorporation in adaptive learning procedures and development of inversion schemes.

Significant progress in this area was achieved during the recent few years. New methods that were developed include the truncated T-matrix approach,¹ variational methods,² the Distorted Wave Born Approximation³ (DWBA) and extensions of geometrical diffraction techniques.⁴ Comparison of results obtained using different techniques and also with experiment are most encouraging. However, all techniques have limitations and different regions of validity.

In the DWBA a general shaped defect is represented as a sphere and a remainder volume R which is treated as a perturbation. In some cases this approach gave surprisingly good results even for large volume perturbations. However, in the case of strong scatterers (such as cavities), the DWBA will break down when the remainder volume R is large. To investigate the limitations of the DWBA prior to embarking on data inversion based on it, we studied scattering by a sequence of defects with increasing R . The results of this investigation are presented in Section II.

Obviously, the DWBA has quite general limitations. In many practical situations it is not possible to present a defect as a sphere and a small volume perturbation. Two such defects are shown in Fig. 1. The defect of Fig. 1a consists of a sphere with a circular crack emanating from its equator. Since the crack is not a volume perturbation, its contribution to the scattering can not be evaluated by means of the DWBA. Figure 1b presents a compound defect that consists of two

closely spaced spheres of slightly differing size. When one sphere is much smaller than the other, the DWBA is expected to yield good results. However, when the sizes are about the same, there is no justification to treat one exactly and the other as a perturbation. This multiple defect is the prototype of situations in bond areas, where one has to deal with an aggregate of equally important volume defects. The aim of our investigation is to develop a method that can provide reasonable approximations to the scattering from defects such as those of Fig. 1a and 1b. The basis of the method is to represent the scatterer as two parts R_1 and R_2 , such that the exact or approximate solutions to both problems (R_1 present only, and R_2 only) are known. Then we apply a method of multiple scattering, treating the interaction between the scatterers as perturbation.

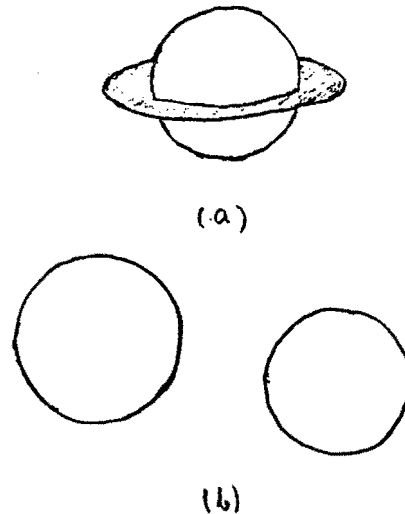


Fig. 1 Two complex defects (a) circular crack emanating from spherical cavity, and (b) two spherical cavities.

This multiple scattering formalism is developed in Section III, and applied for specific geometries in Section IV.

SCATTERING BY NONSPHERICAL DEFECTS - DWBA

The formalism of the Distorted Wave Born Approximation was presented elsewhere;³ here we only summarize the results. Representing the defect as a sphere S and a remainder volume R , the scattered longitudinal wave in direction r , for an incident longitudinal wave along r_0 is given by

$$u_L = u_L^S + \hat{r}\alpha^2(D_1 + D_2 + D_3)\exp(i\alpha r)/r$$

where α is the longitudinal wave vector and the coefficients D_i are expressed in terms of the known solutions of the spherical scattering problem, integrated over the volume R . We considered a sequence of defects as shown in Fig. 2. This sequence represents increasing deviations from spherical shape. The defects considered consist of a large spherical cavity (radius 400μ) and a small hemisphere (of radius 200μ), connected by a cylindrical "neck" of length l . We studied the cases $l = 0, 100, 200, 400 \mu$. The scattered power was calculated, for illuminations from back, side, and front of the small hemisphere. Both angular distribution of the scattered power at fixed frequencies and the frequency dependence of the backscattered power were calculated. Some of the results of these calculations are shown in Figs. 3, 4, 5. All three figures show the angular dependence of scattered power for $\alpha a = 2$. It should be noted that for $\alpha a = 1$ deviations from the pattern obtained for a sphere (the solid line of Fig. 3, 4) are too small to measure, even for the defect with $l = 400$. Deviation from spherical shape is most noticeable for illumination from the side, i.e., a direction of low symmetry (Fig. 4). This fact is quite important for quantitative defect characterization. Consider now Fig. 5, which compares scattered power for $\alpha a = 2, \theta = 135^\circ$, as a function of the azimuthal angle ϕ , for a sequence of defects. A spherical defect would yield constant scattered power; deviations from spherical shape are, as expected, more pronounced for the higher l values. We have also studied backscattered power as a function of frequency, for various directions of incidence and scatterer shapes. However, to our disappointment we found, that backscattered longitudinal power (a relatively simple measurement) is not sensitive enough to the detailed shape of the scatterer. More information can apparently be obtained from analysis of experiments in which angular dependence of the scattered power is measured. Again we observed that deviations from spherical shape are hardly noticeable for $\alpha a < 1.5$.

Detailed comparison of our results with experiment and other methods is given by Tittmann in these proceedings.⁸

MULTIPLE SCATTERING THEORY - GENERAL

In this Section we present a systematic expansion for scattering by two defects. An alternative approach was taken by Varadan et al⁹ to the problem of scattering by two spheres. Their work is based on the truncated T matrix approximation. Although limited to problems with

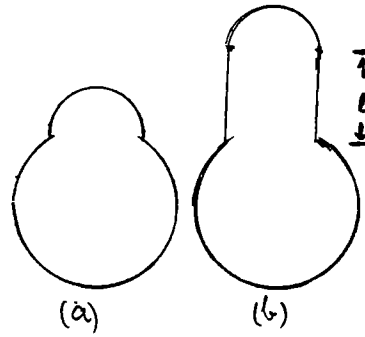


Fig. 2 A sequence of non spherical defects; a small hemispherical cavity is connected to a large spherical one by a "neck" of length l . As l increases from zero, the scatterer deviates more from spherical shape.

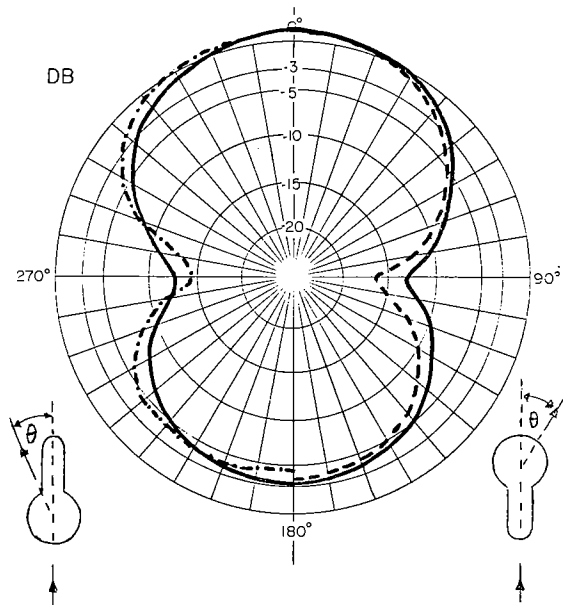


Fig. 3 Scattered power as a function of scattering angle for two directions of incidence, on defect with $l = 400 \mu$. The right half corresponds to the situation with the small hemisphere directly illuminated; the left hand side to incidence on the large sphere. $\alpha a = 2$, is the incident wave number and a the large sphere radius, $a = 400 \mu$.

cylindrical symmetry, their method is powerful and can yield approximate solutions of increasing accuracy to the particular problem studied. Our method is easily generalizable for more than two defects and no symmetry requirement is imposed.

Consider an infinite elastic medium characterized by density ρ_0 and elastic tensor C_{ijkl} , in which two regions R_1 and R_2 with densities ρ_1, ρ_2 and elastic tensors C^1 and C^2 are embedded. We look for solutions of the wave equation

$$(C_{ijkl}u_{k,l})_{,j} + \rho\omega^2u_i = 0 \quad (1)$$

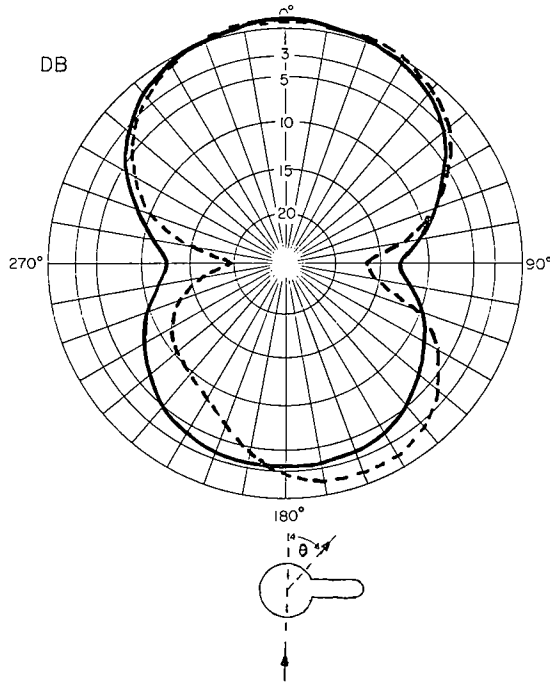


Fig. 4 Angular dependence of scattered power for incidence from the side; $a = 2$, $a = 1 = 400 \mu$.

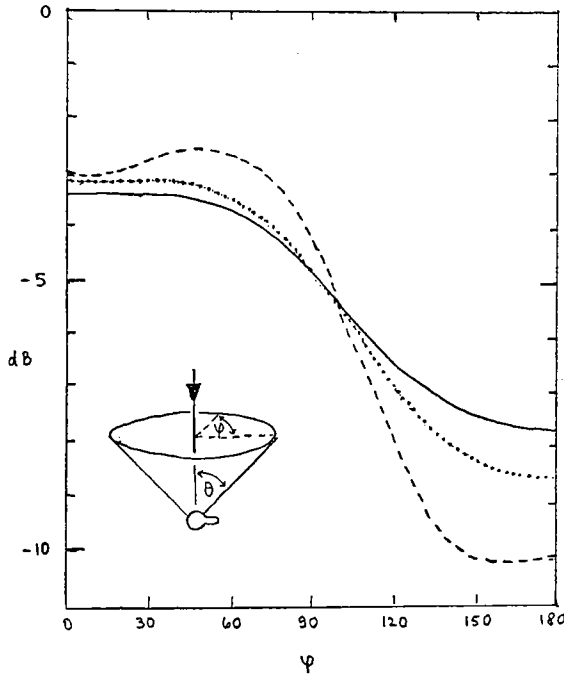


Fig. 5 Power scattered at $\theta = 135^\circ$, as a function of azimuthal angle ρ , for incidence from the side on defects with $l = 0$ (solid line) $l = 200 \mu$ (dotted line) and $l = 400 \mu$ (dashed line).

of the form

$$u_i(r) = u_i^0(r) + u_i^S(r),$$

where u^0 is a plane wave and u^S an outgoing spherical wave.

Using the formalism developed by Gubernatis et al⁵ the solution of (1) can be written as

$$u_i(r) = u_i^0(r) + \sum_{\alpha=1,2} \int dr' [g^0(r, r') V^{(\alpha)} u(r')] \quad (2)$$

where we used the notation

$$[gV^{(\alpha)} u]_i = [\delta \rho^{(\alpha)} \omega^2 g_{im}^0 u_m - \delta C_{jklm}^{(\alpha)} g_{ij,k}^0 u_{l,m}] \theta_{R_\alpha}(r) \quad (3)$$

with

$$\begin{aligned} \delta \rho^{(\alpha)} &= \rho_\alpha - \rho_0 \\ \delta C^{(\alpha)} &= C^{(\alpha)} - C^0 \end{aligned}$$

and

$$\theta_{R_\alpha}(r) = \begin{cases} 1 & r \in R_\alpha \\ 0 & \text{otherwise} \end{cases}$$

and $g^0(r, r')$ is the infinite medium Greens function.

Equation (2) is a generalization of the integral equation for scattering by a single defect. From now on we will suppress all tensorial indices and integrations; thus Eq. (2) is rewritten as

$$u = u^0 + g^0 V^{(1)} u + g^0 V^{(2)} u \quad (4)$$

It is useful to introduce the solutions to the scattering problems with each defect (i.e., when only α is present);

$$u^{(\alpha)} = u^0 + g^0 V^{(\alpha)} u^{(\alpha)} = u^0 + u^{(\alpha)} S, \alpha = 1, 2 \quad (5)$$

and the respective Greens functions that satisfy the equations

$$g^{(\alpha)} = g^0 + g^0 V^{(\alpha)} g^{(\alpha)} = g^0 + g^{(\alpha)} S, \alpha = 1, 2 \quad (6)$$

Equations (4)-(6) are integral equations; the (unknown) solution appears on both sides. To express the solution in terms of the incident wave only, one introduces the T matrix. For a single scatterer problem $T^{(\alpha)}$ is defined by

$$u^{(\alpha)} = u^0 + g^0 T^{(\alpha)} u^0 \quad (7)$$

and for the problem with both scatterers present one has

$$u = u^0 + g^0 T u^0$$

By generating the infinite Born series using (4) and performing partial summations, one obtains an expansion for T in terms of $T^{(1)}$ and $T^{(2)}$;

$$T = T^{(1)} + T^{(2)} + T^{(1)}g^0T^{(2)} + T^{(2)}g^0T^{(1)} + O(T^{(\alpha)})^3 \quad (8)$$

In the approximation adopted here we neglect all higher than second order terms in the $T^{(\alpha)}$. Physically, our approximation contains the following processes: scattering from R_1 and from R_2 (single scattering processes); scattering from R_1 first and then from R_2 and vice versa (double scattering processes). The first two terms in (8) contribute to the full solution u the coherent sum of the two separate scattering problems; the next two terms constitute the lowest order approximation to the interaction between the scatterers. Using the approximation (8) for the T matrix in Eq. (7), the solution u is expressed as

$$u = u^0 + g^{(1)}V^{(2)}u^{(2)} + g^{(2)}V^{(1)}u^{(1)} \quad (9)$$

Alternatively, this can be rewritten as

$$u = u^0 + u^{(1)}S_+ + u^{(2)}S_+ + g^{(1)}S_V^{(2)}u^{(2)} + g^{(2)}S_V^{(1)}u^{(1)} \quad (10)$$

On these equations are based our approximate solutions for scattering by the defects of Fig. 1. Application to the defect of Fig. 1b with two spherical elastic inclusions is straightforward; the solutions to scattering by a sphere are known,⁶ and the spherical Greens functions were expressed in terms of these solutions.³ As to the defect of Fig. 1a, one needs an approximate solution for scattering by a circular annulus and a generalization of the previous formalism for non-volume defects. Such a generalization is needed for the two sphere defect also, when the two spheres are cavities. This is so since our expression for u involves the exact solution for sphere (1) inside sphere (1) itself, where it is not defined. Therefore in order to use Eq. (9) or (10) for cavities, a formulation of multiple scattering theory on the basis of surface integrals is needed.

Surface Formalism.⁷ The starting point is the integral equation for the solution, in terms of integrals over the entire surface of the defect, S .

$$u_m = u_m^0 + C_{ijkl}^0 \int_S ds' \hat{n}_i [u_j g_{mk}^0, 1] - g_{mj}^0 u_{k,1} \quad (11)$$

It should be noted that for the case of a cavity (or crack) the normal stress on the surface S vanishes and only the first term in (11) is needed.

This integral equation can be expressed in operator form

$$u_m = u_m^0 + \int_S ds' n_i g_{mk}^0 V_{ijkl} u_j \quad (12)$$

where

$$V_{ijkl} = C_{ijkl} \hat{\delta}_{l1} - C_{ijkl} \hat{\delta}_{l1} \quad (13)$$

The arrow on the partial derivative indicates whether it acts on g^0 or u . As above, we will suppress the integration and the tensorial indices to write

$$u = u^0 + g^0 V u$$

When the surface of the compound defect is the union of S_1 and S_2 , the surfaces of two defects with known solutions, the formalism of part A can be directly applied, again yielding expressions (9) or (10) as the approximate solutions for the compound defect.

APPLICATION TO TWO SPECIFIC DEFECTS

Two Spherical Cavities. The exact solutions for scattering of plane waves by a single spherical cavity are known. We have also been able to express the Green's function associated with a spherical defect in terms of these solutions. The following notation is used: $u^{(\alpha)}$ is the solution associated with sphere α only, and

$$\sigma_{ij}^{(\alpha)} = (u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)})/2 \quad (14)$$

$$\tau_{ij}^{(\alpha)} = (\hat{n}_i u_{j,j}^{(\alpha)} + \hat{n}_j u_{i,i}^{(\alpha)})/2$$

We use the convention of Ref. (3), with $u_r(r, k, \epsilon)$ denoting the full solution of the scattering problem at point r , for an incident plane wave with wave vector k and polarization ϵ . The displacement field due to two spherical cavities $\alpha = 1, 2$ is given by

$$u_m(r) = u_m^0(r, -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) + \sum_{\epsilon} \hat{e}^{\epsilon} \gamma(\epsilon)^2 \exp[i\gamma(\epsilon)r]/r [\tilde{D}_2(\epsilon) + \tilde{D}_3(\epsilon)] \quad (15)$$

$\gamma(\epsilon)$ is the wave number for longitudinal ($\hat{e}^{\epsilon} = \hat{r}$) or shear ($\hat{e}^{\epsilon} = \hat{\theta}$ or $\hat{\phi}$) waves; the coefficients \tilde{D}_2 and \tilde{D}_3 are given by

$$\tilde{D}_2 = \frac{\lambda}{4\pi\mu\omega^2} \left[\int_{S_1} ds' n_k u_k^{(1)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) u_{1,1}^{(2)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) + \int_{S_2} ds' n_k u_k^{(2)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) u_{1,1}^{(1)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) \right] \quad (16)$$

$$\tilde{D}_3 = \frac{2\mu}{4\pi\omega^2} \left[\int_{S_1} ds' \sigma_{ij}^{(2)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) \tau_{ij}^{(1)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) + \int_{S_2} ds' \sigma_{ij}^{(1)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) \tau_{ij}^{(2)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) \right] \quad (17)$$

where λ, μ are the Lamé parameters and ρ the density of the host medium.

Crack Emanating from Volume Defect. The solution for the scattering problem is again expressed in terms of the solutions for the two separate parts of the defect. In this case, these are a spherical cavity and a circular annulus of vanishing volume, on the surface of which the normal stress

vanishes. For the spherical cavity the exact solution is known; however, this is not so for the annulus. Therefore one needs an estimate for the scattering from the latter defect. We propose the following approximate solution; use a reliable approximation for scattering by a circular crack to obtain the crack opening (i.e., the discontinuity Δu_i on the surface of the crack) and assume that this function represents well the discontinuity for the annulus, even when the central part of the crack is replaced by appropriate boundary conditions on the annulus. Under this assumption one obtains for the scattering from the compound defect the following expression:

$$u_m(r) = u_m^0(r, -\gamma \hat{r}_0, \epsilon_0) + \sum_{\epsilon} \hat{e}^{\epsilon} \gamma(\epsilon)^2 \exp[i\gamma(\epsilon)r] / r [D_2^1(\epsilon) + D_3^1(3)] \quad (18)$$

where $u^{(1)}$ is the solution of the scattering problem from the spherical cavity, $u^{(2)}$ from the circular annulus. This function is expressed as an integral over the discontinuity of the displacement field across a circular crack, $u^{(2)}$, which also enters in the formulae for D_2^1 , D_3^1 :

$$D_2^1 = \frac{\lambda}{4\pi\rho\omega^2} \left[\int_{S_1} ds' n_j u_j^{(1)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) u_{i,j}^{(2)}(r', -\gamma(\epsilon)r, \epsilon) + \int_{S_2} ds' n_j \Delta u_j^{(2)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) u_{i,j}^{(1)}(r', -\gamma(\epsilon)r, \epsilon) \right] \quad (19)$$

$$D_3^1 = \frac{2}{4\pi\rho\omega^2} \left[\int_{S_1} ds' \tau_{ij}^{(1)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) \sigma_{ij}^{(2)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) + \int_{S_2} ds' \sigma_{ij}^{(1)}(r', -\gamma(\epsilon)\hat{r}, \epsilon) \Delta \tau_{ij}^{(2)}(r', -\gamma(\epsilon_0)\hat{r}_0, \epsilon_0) \right] \quad (20)$$

Numerical codes for the approximate solutions based on Eqs. (17)-(20) are currently being developed.

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SUMMARY DISCUSSION

Bruce Thompson, Chairman (Rockwell Science Center [now Ames Laboratory]): There is time for one brief question.

Unidentified Speaker: John Simmons (NBS) has just completed a general variational method for dealing with statistical distributions of defects in multiple scattering, and he claims it's the most exact variation.

Eytan Domany (Weizmann Institute of Science): It's a variational method.

Bruce Thompson, Chairman: Thank you, Eytan, we will now move on to the next paper.